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3. Lattice vibration

5. Phase and group velocities

5.1 Group velocity

Beat is a periodic variation in amplitude as a result of interference between two waves of slightly different frequencies. Beat is also referred to as wave packet. The superposition of two waves which have slightly different frequencies is shown in Figure 1. The horizontal axis of Figure 1 shows time, while the vertical axis shows the water surface elevation. The superposition of these two waves shows such modulation of amplitude. Such superposition is clearly shown in a youtube video (https://www.youtube.com/watch?v=5yiCA5eAcus&t=0s), and Figure 2 is a scene cut from the video. In the video, we can see a red wave and a blue wave. The two waves show superposition resulting in the modulation of amplitude, and this amplitude propagates in the x-direction. The velocity of the propagation of the periodic amplitude variation is referred to as group velocity (v_g), and the velocity of the phase propagation of each wave is referred to as phase velocity (v_p).

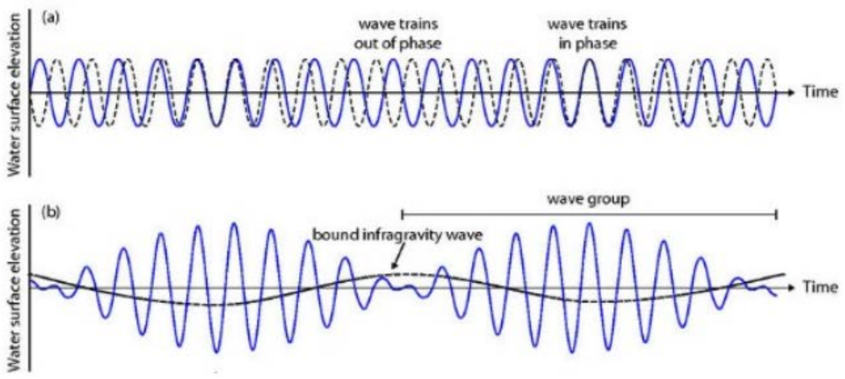

Fig. 1. Schematic diagram of showing the superposition of two waves which have slightly different frequencies. a) Time evolution of amplitudes of the two waves. b) Superposition of the two waves shown in Figure 1a.

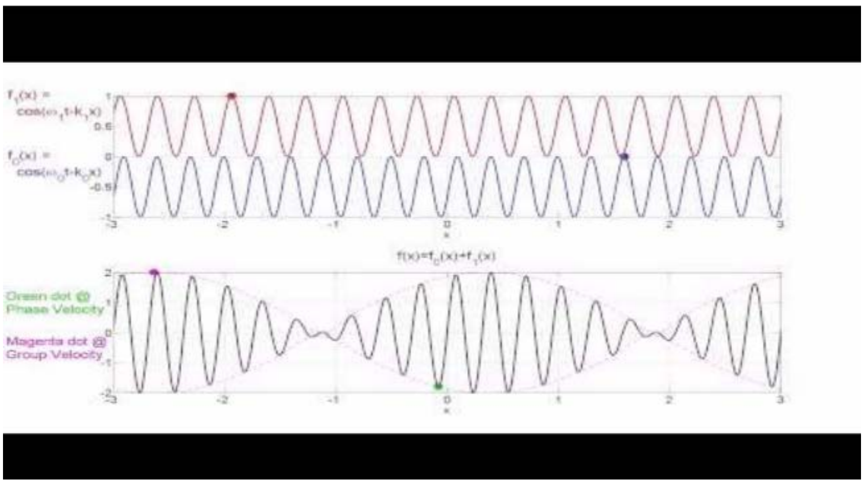

Fig. 2. A scene from a youtube video (https://www.youtube.com/watch?v=5yiCA5eAcus&t=0s) which explains the group velocity and phase velocity. The red wave and the blue wave have slightly different frequencies.

Superposition of two waves is expressed using the formula of a wave. Here, we consider two waves (f_1 and f_2) expressed with the formulas below:

$$f_1(x,t) = exp[i(k_1x - \omega_1 t)],\quad f_2(x,t) = exp[i(k_2x - \omega_2 t)] \tag{3.1.1}$$

where k_1 and k_2 are angular wave numbers while ω_1 and ω_2 are angular frequencies of f_1 and f_2. In the formulas, amplitudes are omitted. (According to the lecture material, "amplitudes are not omitted here", but I wonder amplitudes might be omitted here.) Adding up the formulas of the two waves, wave (f) which is produced by the superposition becomes

$$\begin{aligned} f(x,t) &= f_1(x,t) + f_2(x,t) \\ &= \{cos[(k_1x - \omega_1 t)] + isin[(k_1x - \omega_1 t)]\} \\ &+ \{cos[(k_2x - \omega_2 t)] + isin[(k_2x - \omega_2 t)]\} \\ &= 2cos\left[\frac{(k_1x - \omega_1 t) + (k_2x - \omega_2 t)}{2}\right] cos\left[\frac{(k_1x - \omega_1 t) - (k_2x - \omega_2 t)}{2}\right] \\ &+ 2isin\left[\frac{(k_1x - \omega_1 t) + (k_2x - \omega_2 t)}{2}\right] cos\left[\frac{(k_1x - \omega_1 t) - (k_2x - \omega_2 t)}{2}\right] \\ &= 2cos\left[\frac{(k_1x - \omega_1 t) - (k_2x - \omega_2 t)}{2}\right] exp\left[i\frac{(k_1x - \omega_1 t) + (k_2x - \omega_2 t)}{2}\right] \end{aligned} \tag{3.1.2}$$

using the addition theorem of trigonometric functions. Because the factor of the cosine function in the formula (3.1.2) is the real part, this shows beat, or amplitude modulation, in other words. On the other hand, the complex exponential function in the formula (3.1.2) is the imaginary part, this shows phase propagation.

Next, we will take a closer look at beat (f_b (x, t)), which is amplitude variation. Beat is expressed as

$$\begin{aligned} f_b(x,t) &= cos\left[\frac{(k_1x - \omega_1 t) - (k_2x - \omega_2 t)}{2}\right] \\ &= cos\left[\frac{\Delta k \cdot x - \Delta\omega \cdot t}{2}\right] \cong cos\left[\frac{\Delta k \cdot x - \frac{d\omega}{dk}\Delta k \cdot t}{2}\right] \\ &= cos\left[\frac{\Delta k}{2}\left(x - \frac{d\omega}{dk}t\right)\right] \end{aligned} \tag{3.1.3}$$

where $\Delta k = k_1 - k_2$, and $\Delta\omega = \omega_1 - \omega_2$, as introduced in the formula (3.1.2). From the formula (3.1.3), the velocity of the beat, i. e., the group velocity (v_g), becomes

$$v_g = \frac{d\omega}{dk} \tag{3.1.4}$$

where dω is differential of angular frequency and dk is differential of angular wave number. The group velocity (v_g) is explained in Figure 3. In Figure 3, green curves and arrow show beat propagating with the group velocity (v_g).

5.2 Phase velocity

In this section, we will consider the phase velocity (v_p). From the equation (3.1.2), the oscillation within the beat (f_p (x, t)) is

$$\begin{aligned} f_p(x,t) &= exp\left[i\frac{(k_1x - \omega_1 t) + (k_2x - \omega_2 t)}{2}\right] \\ &= exp\left[i\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\right] \end{aligned}$$

$$= exp\left[i\frac{k_1+k_2}{2}\left(x-\frac{\omega_1+\omega_2}{k_1+k_2}t\right)\right]$$

. From this formula, the phase velocity (v_p) is

$$v_p=\frac{(\omega_1+\omega_2)/2}{(k_1+k_2)/2}=\frac{\overline{\omega}}{\overline{k}} \quad (3.1.5)$$

where $\overline{\omega}$ is average angular frequency and $\overline{k}$ is average angular wave number. The group velocity (v_p) is explained in Figure 3. In Figure 3, blue curves and arrow show that phase of inside of the beat independently propagates with the phase velocity (v_p).

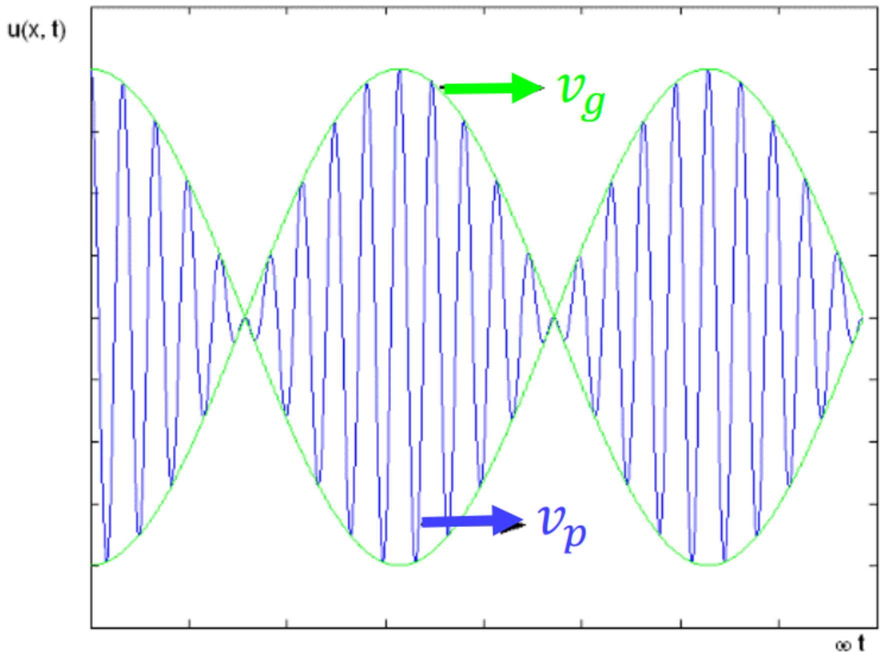

Fig. 3. A diagram which shows a wave propagation. A wave packet propagates with the group velocity (v_g) and the phase velocity (v_p). The green curves show beat, and the blue curve shows phase of inside of the beat. Note that the phase propagates independently of the beat.

5.3 Dispersion relation

In this section, we will consider the ratio between angular frequency (ω) and angular wave number (k). The ratio is referred to as dispersion relation, and expressed as

$$\omega=f(k) \quad (3.1.6)$$

since ω is a function of k. This relation is expressed as the red curve in Figure 4, provided that the ratio is not constant. Dispersion relation is related to group velocity (v_g) and phase velocity (v_p). As explained in the previous sections, group velocity (v_g) is

$$v_g=\frac{d\omega}{dk} \quad (3.1.4)$$

and phase velocity (v_p) is

$$v_p=\frac{\omega}{k} \quad (3.1.5)$$

. Therefore, v_g is equal to differential of ω with respect to differential of k, which means that the slope of the curve of the dispersion relation shows v_g. On the other hand, v_p is the ratio between ω and k, which means that the ratio between ω and k at each point of the curve of the dispersion relation shows v_p.

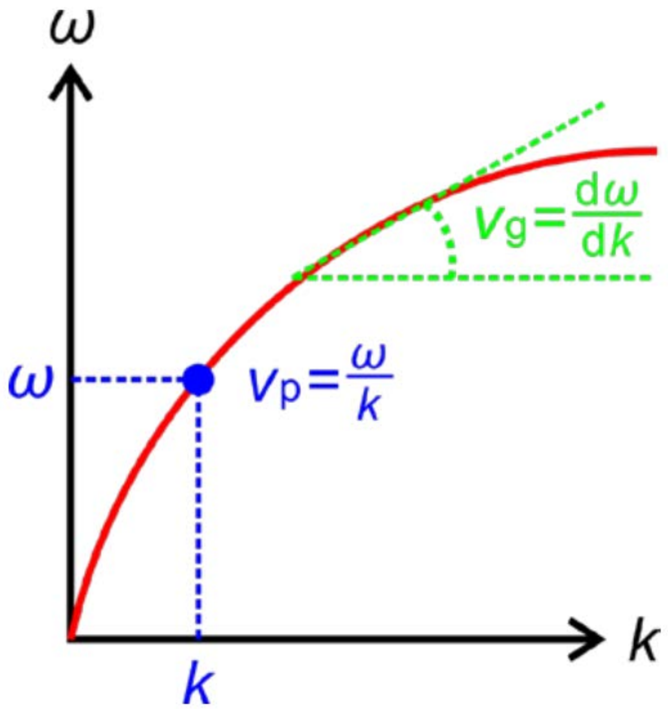

Fig. 4. The relation between angular frequency (ω) and angular wave number (k), that is, dispersion relation. The red curve shows the dispersion relation. The green broken lines denote group velocity (v_g) while the blue broken lines denote phase velocity (v_p).