Canberra, 29 December 2021

9. Acoustic Impedance

1. What is acoustic impedance?

Acoustic impedance Z_s is the product of density ρ and seismic velocity υ of a rock unit and it is an essential parameter for judging the reflectivity and transmissivity of seismic waves

$$Z_s = \rho\upsilon \qquad (9.2.1.)$$

Mineral physics is the science of materials that compose the interior of planets, such as Earth, and provides information that can provide insights into plate tectonics, mantle convection, the geodynamo and related phenomena. Mineral physicists provide information about material properties in Earth's deep interior. This permits the interpretation of surface measurements of seismic waves, gravity anomalies, geomagnetic fields, and electromagnetic fields.

The work of mineral physicists overlaps with rock physics – the study of the relationship between physical and elastic properties of rocks. Rock physics is active at variable scales of investigation by researcher of different backgrounds, from seismic data interpretation by geophysicists to petrophysicists.

The interpretation of seismic data provides information about the Earth's interior, among many other uses. The Earth's structure is discontinuous (Fig. 3). When seismic waves travel through the Earth's interior, they are reflected and / or transmitted at those discontinuities, for example the Mohorovičić discontinuity, the 660-km discontinuity, and the core-mantle boundary.

Because acoustic impedance depends on the density and seismic velocity of a rock unit, different rock units have different acoustic impedance. These acoustic impedance contrasts at discontinuities within the Earth's interior influence the reflection and transmission of seismic waves. Therefore, in this chapter, we will learn about acoustic impedance by studying its influence on seismic waves moving through the Earth's interior.

2. Energy of 1D waves

The total energy U_T of a sinusoidal, elastic, one-dimensional wave is the sum of the potential energy (strain) U_p and the kinetic energy U_K:

$$U_p = \frac{1}{2} * E\varepsilon^2$$

$$U_K = \frac{1}{2} * \rho u^2$$

Where E is Young's modulus, ε is strain, ρ is density and u is wave amplitude.

In a continuous body, such as the Earth, a deformation field results from a stress field due to applied forces. Stress and strain are related by constitutive equations, such as Hooke's law. Elastic deformation is deformation, which ceases to exist after the stress field is removed. Thus, potential (strain) energy is the energy held by an object due to stresses acting on it.

The kinetic energy of an object is the energy that it possesses due to its motion and is defined as the work needed to accelerate a body of a given mass from rest to its stated velocity. This kinetic energy is maintained unless the object's speed changes.

Seismic waves are energy waves that travel through the interior of planets, such as Earth. Natural causes of seismic waves are earthquakes, volcanic eruptions, magma movement and large landslides. Seismic waves can also be the result of large man-made explosions that give out low-frequency acoustic energy.

The wave equation is used to describe mechanical waves, such as seismic waves, and solving the wave equation provides information on how waves are reflected and / or transmitted at the boundary between two media. In this chapter, we focus on the scalar wave equation describing waves in scalars by scalar functions of a time variable *t* and a spatial variable *x*. The real part of the wave equation is written as:

$$u = u_0 \cos(kx - \omega t) \tag{9.2.2.}$$

$$\frac{\delta u}{\delta x} = -k\, u_0 \sin(kx - \omega t) \tag{9.2.3.}$$

$$\frac{\delta u}{\delta t} = \omega\, u_0 \sin(kx - \omega t) \tag{9.2.4.}$$

Substituting the real part of the wave equation into the formulas for potential energy and kinetic energy results in the following formulas:

$$u_p = \frac{Ek^2{u_0}^2}{2}\frac{1-\cos 2(kx - \omega t)}{2} \tag{9.2.5.}$$

$$u_k = \frac{\rho\omega^2{u_0}^2}{2}\,\frac{1-\cos 2(kx - \omega t)}{2} \tag{9.2.6.}$$

Thus, the total energy of a one-dimensional wave (9.2.7.) can be written as

$$u_T = u_p + u_K \tag{9.2.7.}$$

$$\begin{aligned} u_T &= \frac{Ek^2{u_0}^2}{2}\frac{1-\cos 2(kx - \omega t)}{2} + \frac{\rho\omega^2{u_0}^2}{2}\,\frac{1-\cos 2(kx - \omega t)}{2} \\ &= \frac{1}{2}[Ek^2 + \rho\omega^2]{u_0}^2\frac{1-\cos 2(kx - \omega t)}{2} \\ &= \frac{1}{2}\left[\frac{E}{\rho}\left(\frac{k}{\omega}\right)^2 + 1\right]\rho\omega^2{u_0}^2\frac{1-\cos 2(kx - \omega t)}{2} \\ &= \frac{1}{2}\left[\upsilon^2\left(\frac{1}{\upsilon}\right)^2 + 1\right]\rho\omega^2{u_0}^2\frac{1-\cos 2(kx - \omega t)}{2} \\ &= \rho\omega^2{u_0}^2\frac{1-\cos 2(kx - \omega t)}{2} \end{aligned} \tag{9.2.8.}$$

because $\upsilon = f\lambda = \frac{\omega}{k}$ and $\upsilon = \frac{\sqrt{E}}{\rho}$, where λ is the wavelength, u is the aplitude, ω is the angular frequency, k is the wave number and v is the velocity.

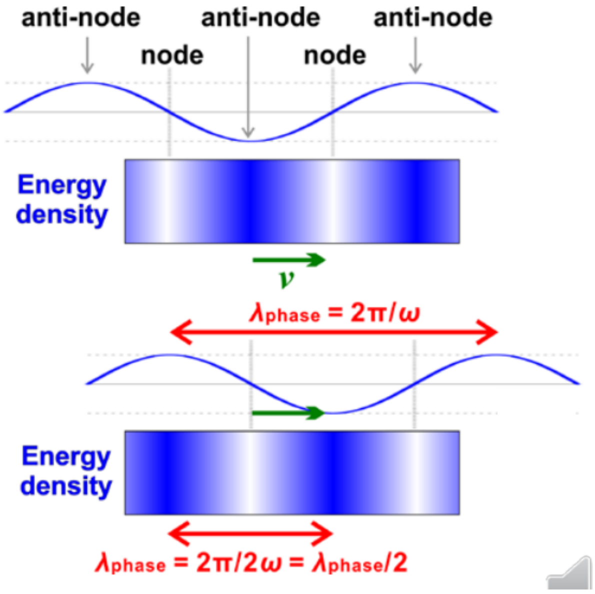


Fig. 1. Conceptual diagram of a propagating wave. The blue line represents the wave, and the blue-white bar shows the energy density.

This shows that the average energy of a wave is proportional to ρ, ω^2 and u_0^2.

$u_T \propto \frac{1-\cos 2(kx-\omega t)}{2}$, which means that it is not kept constant within a given volume element. Figure 1 shows that:

The energy maxima are located at the anti-node, i.e., at $\cos(kx - \omega t) = \pm 1$, and

The energy minima (zero) are located at the node, i.e., at $\cos(kx - \omega t) = 0$.

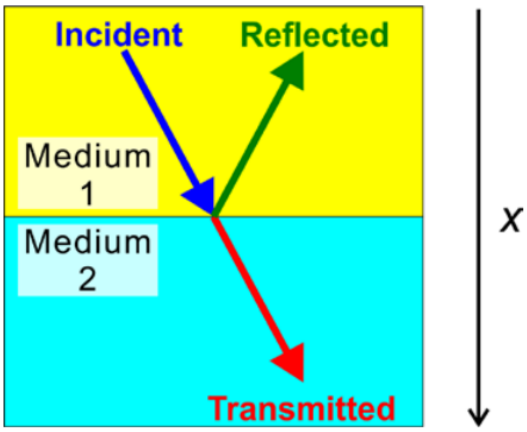
Energy is transferred by movement of nodes. Transferred energy per unit time is proportional to the wave velocity:

$$\rho\omega^2 {u_0}^2 * v \qquad (9.2.9.)$$

3. Reflection and transmission

When a seismic wave encounters a boundary between two materials with different acoustic impedances, a part of the energy in the wave will be reflected at the boundary, while the other part of the energy in the wave will be transmitted through the boundary. The total energy of the transmitted and reflected waves must be equal to the energy of the incident wave. The amount of energy transmitted through the interface (discontinuity) is inversely proportional to the acoustic impedance. This means, that the smaller the contrast in acoustic impedance across the discontinuity, the greater is the portion of transmitted energy. Conversely, the greater the contrast in acoustic impedance across the discontinuity, the more energy is reflected at the interface. The relative amplitudes of the transmitted and reflected waves depend on the velocities, densities, and the angle of incidence.

In this setting an incident wave from Medium 1 is transmitted to Medium 2 and reflected at the boundary in Medium 1 (Fig. 2). Medium 1 and Medium 2 are characterized by different velocities, which leads to different acoustic impedances according to formula 9.2.1. Because of this, part of the incident wave is reflected at the boundary between Medium 1 and 2 and the other part is transmitted through the boundary.



- Notation:
  - $u_i(x,t)$: the displacement by the incident wave at point x and time t
  - $u_r(x,t)$: the displacement by the reflected wave at point x and time t
  - $u_t(x,t)$: the displacement by the transmitted wave at point x and time t
  - u_{i0}: the amplitude of the incident wave
  - u_{r0}: the amplitude of the reflected wave
  - u_{t0}: the amplitude of the transmitted wave
  - v_1: the velocity in Medium 1
  - v_2: the velocity in Medium 2

Fig. 2: Setting of reflection and transmission of a wave and the downward direction is +x (left). Notation of parameters used for solving the wave equation (right).

The displacements of both media at the boundary must be equal, so in this example (Fig. 2), the total displacement is the sum of the displacement of the incident and reflected waves at time t and point x, which is equal to the displacement of the transmitted wave at time t and point x:

$$u_{i(0,t)} + u_{r(0,t)} = u_{t(0,t)}$$

Which, at time = 0 and point = 0, is expressed as:

$$u_{i0} + u_{r0} = u_{t0} \qquad (9.3.10.)$$

The law of conservation of energy states that the total energy of an isolated system remains constant, i.e., it is conserved over time. The *conservation of energy* is expressed as:

$$\rho_1 \omega^2 {u_{i0}}^2 \upsilon_1 = \rho_1 \omega^2 {u_{r0}}^2 \upsilon_1 + \rho_2 \omega^2 {u_{t0}}^2 \upsilon_2 \qquad (9.3.11.)$$

By substituting (9.3.9.) into (9.3.10.) and rearranging the equation so one side equals zero, we get

$$\rho_1 \upsilon_1 {u_{i0}}^2 - \rho_1 \upsilon_1 {u_{r0}}^2 - \rho_2 \upsilon_2 {u_{i0}}^2 - 2\ \rho_2 \upsilon_2 u_{i0} u_{r0} - \rho_2 \upsilon_2 {u_{r0}}^2 = 0 \qquad (9.3.12.)$$

Simplifying equation (9.3.11.) results in equation (9.3.12.):

$$[u_{i0} + u_{r0}] * [(\rho_1 \upsilon_1 - \rho_2 \upsilon_2) u_{i0} - (\rho_1 \upsilon_1 + \rho_2 \upsilon_2) u_{r0}] = 0 \qquad (9.3.13.)$$

This equation has two solutions. In solution 1 the term $u_{i0} + u_{r0} = 0$.

In solution 2 the term $[(\rho_1 \upsilon_1 - \rho_2 \upsilon_2) u_{i0} - (\rho_1 \upsilon_1 + \rho_2 \upsilon_2) u_{r0}] = 0$.

Solution 1 describes a case of complete reflection, where no transmission of the wave into Medium 2 occurs (Fig. 3). The reflected wave has the same magnitude of amplitude as the incident wave, but the opposite phase of the incident wave:

$$u_{t0} = u_{i0} + u_{r0} = 0 \qquad (9.3.14.)$$

Solution 2 describes the case where both reflection and transmission occur (Fig. 3):

$$u_{r0} = \frac{\rho_1 \upsilon_1 - \rho_2 \upsilon_2}{\rho_1 \upsilon_1 + \rho_2 \upsilon_2} u_{i0} \qquad (9.3.15.)$$

In this case, the amplitude of the reflected wave is proportional to the difference ρυ between Medium 1 and Medium 2, where $\Delta \rho \upsilon = \rho_1 \upsilon_1 - \rho_2 \upsilon_2$. If Medium 2 is stiffer and heavier $(\rho_1 \upsilon_1 < \rho_2 \upsilon_2)$, anti-phase reflection occurs, where the phase of the reflected wave is opposite to that of the incident wave:

$$u_{t0} = u_{i0} + \frac{\rho_1 \upsilon_1 - \rho_2 \upsilon_2}{\rho_1 \upsilon_1 + \rho_2 \upsilon_2} u_{i0} = \frac{2 \rho_1 \upsilon_1}{\rho_1 \upsilon_1 + \rho_2 \upsilon_2} u_{i0} \qquad (9.3.16.)$$

The incident wave and the transmitted wave have the same phase because ρ_1, υ1, ρ2, υ2 > 0. However, if $\rho_1 \upsilon_1 = \rho_2 \upsilon_2$ and $u_{t0} = u_{i0}$,then complete the incident wave is completely transmitted, and no reflected wave occurs in Medium 1.

4. Acoustic impedance

4.1 Why is Z_s called “impedance”?

Acoustic impedance Z_s is the product of density ρ and seismic velocity υ of a rock unit and it is an essential parameter for judging the reflectivity and transmissivity of seismic waves. When dealing with electricity, impedance is the ratio of voltage to current, in other words, the ratio of the driving force to its results. In mineral physics, acoustic impedance Z_s is the ratio of the stress to the particle velocity and describes the movement of a small element of the medium by stress. Acoustic impedance is calculated using the formulas for particle velocity, the wave function, and stress:

$$v_{particle} = \frac{\delta u}{\delta t} = \omega u0 \mathbf{si\ n}(kx - \omega t) \qquad (9.4.17.)$$

$$u = u_0 \mathbf{co\ s}(kx - \omega t) \qquad (9.4.18.)$$

$$\sigma = E\varepsilon = E \frac{\delta u}{\delta x} = -Eku_0 \mathbf{si\ n}(kx - \omega t) \qquad (9.4.19.)$$

Where $v_{particle}$ is particle velocity, u is wave function and σ is stress. Acoustic impedance Z_s is calculated in the following way:

$$Zs = \left| \frac{\sigma}{v_{particle}} \right| = \frac{Eku_0}{\omega u_0} = \frac{E}{\rho} \frac{k}{\omega} \rho = {v_{phase}}^2 \frac{1}{v_{phase}} \rho = \rho v_{phase} \qquad (9.4.20.)$$

4.2 Reflectivity at the mantle discontinuities

As an example of the effect of acoustic impedance on seismic waves in the Earth's mantle, let us look at the P-wave acoustic impedance contrasts at the 410-km and 660-km discontinuities (Fig. 3).

Many types of surface waves exist, which can be broadly classified as *body waves* and *surface waves*. Body waves travel through the interior of the Earth along paths controlled by material properties – density and Young's modulus (stiffness), which in turn vary according to temperature, material composition and material phase. Surface waves, on the other hand, travel along the Earth's surface. Body waves are distinguished according to types of particle movement into primary (P-) waves and secondary (S-) waves. P-waves are compressional waves that are longitudinal in nature, can travel through any type of material – including fluids, and travel faster than other wave types. Thus, P-waves arrive at seismograph stations first, hence the term "primary" wave.

The amplitude of the reflected wave is predicted by multiplying the amplitude of the incident wave by the seismic reflection coefficient R. The seismic reflection coefficient is determined by the acoustic impedance contrast between the two materials on either side of the discontinuity. R is calculated in the following way:

$$R = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} \tag{9.4.21.}$$

Example 1: **P-wave** acoustic impedance contrasts at **410-km** discontinuity

$$Z_{s,p,410-} = 3.54 \frac{g}{cm^3} * 8.91 \frac{km}{s} = 3.15 * 10^7 \frac{kg}{m^2 s}$$

$$Z_{s,p,410+} = 3.72 \frac{g}{cm^3} * 9.13 \frac{km}{s} = 3.40 * 10^7 \frac{kg}{m^2 s}$$

$$R_{p,400} = \frac{3.15 * 10^7 - 3.40 * 10^7}{3.40 * 10^7 + 3.15 * 10^7} = -3.7\%$$

Example 2: **P-wave** acoustic impedance contrasts at **660-km** discontinuity

$$Z_{s,p,660-} = 3.99 \frac{g}{cm^3} * 10.27 \frac{km}{s} = 4.10 * 10^7 \frac{kg}{m^2 s}$$

$$Z_{s,p,660+} = 4.38 \frac{g}{cm^3} * 10.75 \frac{km}{s} = 4.71 * 10^7 \frac{kg}{m^2 s}$$

$$R_{p,400} = \frac{4.10 * 10^7 - 4.71 * 10^7}{4.71 * 10^7 + 4.10 * 10^7} = -6.9\%$$

This shows that the discontinuity at 660 km is stronger than the one at 410 km. To understand the results of examples (1) and (2), we need to look at phase shift and overturn.

4.3 Phase shift and overturn

During reflection and transmission of seismic waves, *phase shift* and *overturn* can occur. The term *phase shift* describes a different in time between the anti-nodes of two waves. When the time difference between two waves is zero, they are **in phase**. When their signals have the same sign and reinforce each other, it is called constructive interference. In every other case, waves are **out of phase**, and the interference between wave signals in non-constructive.

No phase shift occurs by reflection and transmission when reflectivity and transmissivity are described by real numbers. Likewise, a phase shift occurs by reflection and transmission when reflectivity and transmissivity are described by imaginary numbers.

Overturned waves occur when a downgoing wave is refracted so that it begins propagating upwards. Overturned waves occur because the wave speed of the Earth generally increases with depth. Overturn occurs when reflectivity R is negative, so when the incident wave transitions from a medium with low acoustic impedance Z_s to a medium with high acoustic impedance Z_s. In many cases, this happens when seismic waves move from shallower to deeper zones.

$$\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} < 0 \quad (9.4.22.)$$

No overturn occurs when reflectivity R is positive, so when the incident wave transitions from a medium with high acoustic impedance Z_s to a medium with low acoustic impedance Z_s. In this case, underside reflection of seismic waves occurs. In many cases, this happens when seismic waves move from deeper zones to shallower zones.

$$\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} > 0 \quad (9.4.23.)$$

formula in lecture is $\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} < 0$

Explanation: I changed this sign because no overturn occurs when reflectivity is positive, i.e. greater than 0. In this case, the correct mathematical sign is > instead of <.

Underside reflection occurs when a seismic wave is not reflected while it travels into the interior of the Earth and is reflected on the underside of a discontinuity on while it travels from the Earth's interior to the Earth's surface (Fig. 3).

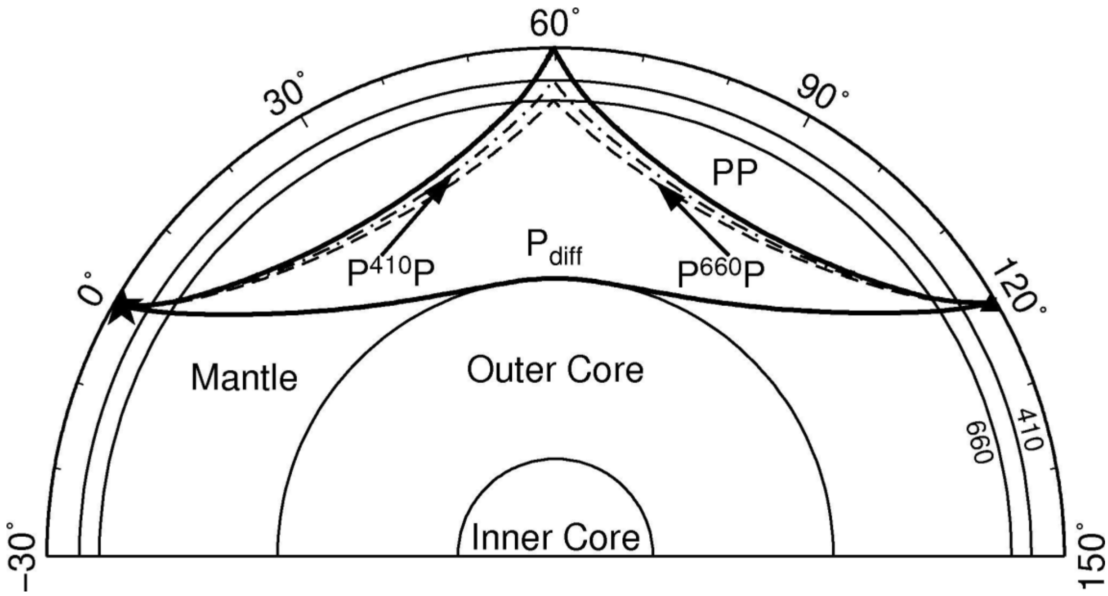


Fig. 3: This schematic diagram illustrates the different paths of seismic waves, which originate at 0° close to the Earth's surface and travel into the interior of the Earth. They encounter different discontinuities, where they are transmitted and / or reflected. The paths termed $P^{410}P$ and $P^{660}P$ illustrate the case of *underside reflection*. This occurs where the seismic waves encounter the 410-km and 660-km discontinuities, respectively.